

OPERATOR MACHINES ON DIRECTED GRAPHS

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ABSTRACT. We show that if an infinite-dimensional Banach space X has a symmetric basis then there exists a bounded, linear operator $R : X \rightarrow X$ such that the set

$$A = \{x \in X : \|R^n(x)\| \rightarrow \infty\}$$

is non-empty and nowhere dense in X . Moreover, if $x \in X \setminus A$ then some subsequence of $(R^n(x))_{n=1}^\infty$ converges weakly to x . This answers in the negative a recent conjecture of Prăjitură. The result can be extended to any Banach space containing an infinite-dimensional, complemented subspace with a symmetric basis; in particular, all ‘classical’ Banach spaces admit such an operator.

1. INTRODUCTION

Given a Banach space X , a bounded linear operator T on X and $x \in X$, we say that the *orbit of x with respect to T* is the set

$$\text{orb}(x, T) = \{T^n(x) : n \geq 0\}.$$

It is well known (cf [8]) that if X is finite-dimensional then $\text{orb}(x, T)$ is ‘regular’, in the sense that either $\|T^n(x)\| \rightarrow 0$, $\|T^n(x)\| \rightarrow \infty$, or there exists $M > 0$ such that $M^{-1} \leq \|T^n(x)\| \leq M$ for all n . The infinite-dimensional situation is very different. Rolewicz provided simple examples of operators on infinite-dimensional spaces which admit *hypercyclic* vectors, that is, vectors with norm-dense orbits.

The study of orbits is connected to the invariant subspace problem. Indeed, an operator T on a Banach space X has no non-trivial, closed, invariant subspaces if and only if $\text{span orb}(x, T)$ is norm-dense for every non-zero $x \in X$.

There is a considerable body of literature on operators which admit hypercyclic vectors. In this note, we study operators with more regular orbits, in particular, those which tend to infinity. This type of orbit has received attention from several authors. For example, in a systematic study of orbits of operators on Hilbert space, Beauzamy provides several sufficient conditions for T to admit a norm-dense set of points x satisfying $\|T^n(x)\| \rightarrow \infty$ [2, Chapter III]. Broadly speaking, these conditions are based on the growth of the sequence $(\|T^n\|)_{n=1}^\infty$. For example, if $\sum_{n=1}^\infty \|T^n\|^{-1} < \infty$ then T admits such a dense set. Sharp estimates of this nature applying to general Banach spaces are given in [6]. We refer the reader to the surveys [4, 5] for additional results on this topic.

Date: May 31, 2009.

2000 Mathematics Subject Classification. 47A05.

Key words and phrases. Orbits of operators.

Both authors are supported by Grant A 100190801 and Institutional Research Plan AV0Z10190503.

A given operator can have both regular and highly irregular orbits, and the exact behaviour of $\text{orb}(x, T)$, as x ranges over X , is not so easy to determine. In [7], Prăjitură makes the following conjecture.

Conjecture 1.1 ([7, Conjecture 2.9]). *Let T be an operator T on a Banach space and let*

$$A_T = \{x \in X : \|T^n(x)\| \rightarrow \infty\}.$$

Then A_T is norm-dense whenever A_T is non-empty.

Of course, if $\|T^n(x)\| \rightarrow \infty$ for some x then $(\|T^n\|)_{n=1}^\infty$ is unbounded, so by the uniform boundedness principle, the set of y with the property that $\sup_n \|T^n(y)\| = \infty$ is a norm-dense G_δ in X . However, this clearly does not say anything about whether $\|T^n(y)\|$ tends to infinity or not. Indeed, the weighted backwards shift operator T on ℓ_p , $1 \leq p < \infty$, given by

$$T(e_i) = \begin{cases} (i/(i-1))^{1/p} e_{i-1} & \text{if } i > 1 \\ 0 & \text{if } i = 1 \end{cases}$$

satisfies $\|T^n\| \rightarrow \infty$, but $\|T^n(x)\| \not\rightarrow \infty$ for all x [6, Example 4]. In [2, pp. 66–68], there is an example of an operator T on Hilbert space satisfying $\|T^n\| \rightarrow \infty$, but $\inf_n \|T^n(x)\| = 0$ for all x .

The object of this note is to show that there is a wide class of Banach spaces which admit operators failing Conjecture 1.1. In fact, by constructing a range of suitable operators, we can impose a reasonable degree of control over the structure of A_T . Clearly, A_T is always radial, in the sense that if $x \in A_T$ and $\lambda \neq 0$ then $\lambda x \in A_T$. Thus we need only trouble ourselves with what happens to points in the unit sphere S_X .

We shall consider both real and complex Banach spaces. Recall that if a Schauder basis $(e_i)_{i=1}^\infty$ of X is symmetric then there exists an equivalent norm $\|\cdot\|$ on X with the property that

$$\left\| \sum_{i=1}^\infty x_i e_i \right\| = \left\| \sum_{i=1}^\infty \lambda_i x_i e_{\pi(i)} \right\|$$

whenever π is a permutation of \mathbb{N} and $|\lambda_i| = 1$ for all i . Such a norm is called *symmetric*; hereafter, whenever we have a Banach space X with a symmetric basis, we shall assume that X is infinite-dimensional and that the associated norm is symmetric. In addition, we shall say that a subset E of a Banach space is *symmetric* if $\lambda x \in E$ whenever $x \in E$ and $|\lambda| = 1$. Here follows our main result.

Theorem 1.2. *Let X have a symmetric basis with norm $\|\cdot\|$ and suppose that $Y \subseteq X$ is a subspace of dimension d , where $2 \leq d < \infty$. Moreover, let $E \subseteq S_Y$ be closed and symmetric, and let J be a projection of X onto Y . Then there exists an operator $R : X \rightarrow X$ with two properties:*

- (1) *if $J(x) \in E$ then $\|R^n(x)\| \rightarrow \infty$;*
- (2) *if $J(x) \in S_Y \setminus E$ then there is a subsequence $(R^{n_i}(x))$ of $(R^n(x))$ such that $R^{n_i}(x) \rightarrow x$ weakly.*

Of course, we obtain the claim in the abstract by ensuring that $E \subseteq S_Y$ in Theorem 1.2 is non-empty and nowhere dense. Roughly speaking, we use the extra dimensions in the complement of Y in X to encode the non-linear information in E . To give an idea of what we mean by this, we can compare, at a distance, this encoding of non-linear information to the standard method of producing an operator on Hilbert space with a prescribed spectrum, namely, by arranging a suitable, countable family of eigenvalues. The proof of Theorem 1.2 is spread across sections 2 and 3. We expect that it is possible to generalise Theorem 1.2 to incorporate subsets of S_Y of greater topological complexity, but to do so would go beyond the immediate aims of this paper and would unduly complicate our existing proof.

Of course, if X in Theorem 1.2 is complemented in some overspace Z then we obtain a corresponding result about Z . In particular, by considering c_0 or ℓ_p , $1 \leq p < \infty$, we can see that any ‘classical’ Banach space admits an operator T such that A_T is non-empty and nowhere dense.

We finish this section by showing that the richness of structure of A_T , demonstrated by Theorem 1.2, cannot be reproduced in the finite-dimensional setting. In fact, we show that the operator R in Theorem 1.2 cannot be compact in general.

Proposition 1.3. *Let X be a Banach space and $T : X \rightarrow X$ a compact operator. Then there exist finite-codimensional subspaces $Z \subseteq Y \subseteq X$ with the property that*

- (1) *if $x \in X \setminus Y$ then $\|T^n(x)\| \rightarrow \infty$;*
- (2) *if $x \in Y \setminus Z$ then there exists $M > 0$ such that $M^{-1} \leq \|T^n(x)\| \leq M$ for every n ;*
- (3) *if $x \in Z$ then $\|T^n(x)\| \rightarrow 0$.*

In particular, if T is compact then A_T is simply the complement of a finite-codimensional subspace and, in particular, either empty or dense. Proposition 1.3 is a mild elaboration and generalisation of the behaviour of orbits in finite-dimensional space, stated at the beginning of this note. The result is probably folklore but we sketch a proof of it for completeness.

Sketch proof of Proposition 1.3. First, we assume that X is complex. Using the standard spectral theory of compact operators and the theory of Jordan normal forms, we know that we can find a (possibly empty) sequence of finite-dimensional subspaces X_1, \dots, X_n , and a finite-codimensional subspace Z , all invariant for T , such that

$$X = X_1 \oplus \dots \oplus X_n \oplus Z.$$

Moreover, the subspaces X_i satisfy the following properties.

- (a) each X_i has some basis $e_{i,1}, \dots, e_{i,m_i}$ and associated biorthogonal functionals $f_{i,1}, \dots, f_{i,m_i}$ on X_i , such that
 - (i) $f_{i,k}(T(e_{i,k})) = \lambda_i$ for $1 \leq k \leq m_i$;
 - (ii) $f_{i,k-1}(T(e_{i,k})) = 1$ for $2 \leq k \leq m_i$;
 - (iii) $f_{i,t}(T(e_{i,k})) = 0$ otherwise.
- (b) if x is an eigenvector of T with eigenvalue λ , and $|\lambda| \geq 1$, then $x \in X_i$ and $\lambda = \lambda_i$ for some $i \leq n$.

Define $Y = [e_{i,1}]_{1 \leq i \leq n, |\lambda_i|=1} \oplus Z$, where $[\cdot]$ denotes norm-closed linear span. To prove the proposition, it is sufficient to show that

- (I) $\|T^n(x)\| \rightarrow \infty$ whenever $x \in X_i \setminus Y$;
- (II) $\|T^n(x)\| = \|x\|$ whenever $x \in X_i \cap Y$
- (III) $\|T^n(x)\| \rightarrow 0$ whenever $x \in Z$.

First, take $x \in X_i$ such that $f_{i,j}(x) = 0$ for $j > k$. Using properties (ai)–(aiii), we see that

$$f_{i,k}(T(x)) = \sum_{l=1}^{m_i} f_{i,k}(T(e_{i,l}))f_{i,l}(x) = \lambda_i f_{i,k}(x)$$

and $f_{i,j}(T(x)) = 0$ for $j > k$. Thus, by induction, $f_{i,k}(T^n(x)) = \lambda_i^n f_{i,k}(x)$. Moreover, if $k > 1$ then we calculate

$$f_{i,k-1}(T(x)) = \sum_{l=1}^{m_i} f_{i,k-1}(T(e_{i,l}))f_{i,l}(x) = \lambda_i f_{i,k-1}(x) + f_{i,k}(x)$$

and by a second induction we obtain

$$f_{i,k-1}(T^n(x)) = \lambda_i^n f_{i,k-1}(x) + n\lambda_i^{n-1} f_{i,k}(x).$$

Now consider (I). If $x \notin Y$ then in particular $f_{i,k}(x) \neq 0$ for maximal $k \leq m_i$. There are two cases. If $|\lambda_i| > 1$ then

$$\|f_{i,k}\| \|T^n(x)\| \geq f_{i,k}(T^n(x)) = |\lambda_i|^n |f_{i,k}(x)| \rightarrow \infty.$$

If instead $|\lambda_i| = 1$ then we must have $k \geq 2$, thus

$$\|f_{i,k-1}\| \|T^n(x)\| \geq |f_{i,k-1}(T^n(x))| \geq n|f_{i,k}(x)| - |f_{i,k-1}(x)| \rightarrow \infty.$$

To see (II), note that the results above give $T^n(e_{i,1}) = \lambda_i^n e_{i,1}$ straightaway. For (III), observe that by property (b) above, we have ensured that the restriction S of T to Z has spectral radius $\alpha < 1$. If $x \in Z$ and $\alpha < \beta < 1$ then

$$\|T^n(x)\| = \|S^n(x)\| \leq \|S^n\| \|x\| \leq \beta^n \|x\| \rightarrow 0.$$

for large enough n .

This completes the proof in the complex case. If X is real then we pass to its complexification $X_{\mathbb{C}}$ and consider the compact operator $T_{\mathbb{C}}$, defined by $T_{\mathbb{C}}(x + iy) = Tx + iTy$. \square

2. LOCAL ESTIMATES

Our map R in Theorem 1.2 is going to be a block diagonal operator on X . In this section, we build the template for the operators acting on the blocks and gather together some basic estimates. Let $m, T \in \mathbb{N}$, $\varepsilon > 0$ and $Y = \ell_p^T$, where $4m \leq T$ and $1 \leq p \leq \infty$. Define the operators $S : Y \rightarrow Y$ and $F : \mathbb{R} \rightarrow Y$ by

$$S(y) = (y_T, y_1, \dots, y_{T-1})$$

where $y = (y_1, \dots, y_T)$, and

$$F(a) = (\underbrace{\varepsilon a, \dots, \varepsilon a}_{m \text{ times}}, \underbrace{-\varepsilon a, \dots, -\varepsilon a}_{m \text{ times}}, 0, \dots, 0).$$

In this way, S can be described as a shift operator and F a ‘feed’ operator. Let $R : \mathbb{R} \oplus Y \longrightarrow \mathbb{R} \oplus Y$ be defined by $R(a, y) = (a, S(y) + F(a))$. We are interested in the behaviour of $R^t(a, 0)$ at time $t \in \mathbb{N}$. We can imagine that S drives an airport baggage carousel and F deposits the passengers’ bags onto the moving belt at a fixed set of positions (although some of the bags have ‘negative mass’). The absolute mass of bags deposited depends on the value of the first coordinate. Aided by this analogy, we can see that the result of repeated applications of R to the vector $(a, 0)$ can be viewed as the sum of two bumps: one stationary bump of height εam and base width $2m$, and a moving bump of height $-\varepsilon am$ and base width again $2m$. The moving bump’s motion is periodic, with period T . Let us denote by P the map $(a, y) \mapsto y$.

Lemma 2.1. *Let $1 \leq p \leq \infty$. Firstly, if $m \leq t \leq T - m$ then*

$$(2.1) \quad ||PR^t(a, 0)|| \geq \begin{cases} \left(\frac{2}{p+1}\right)^{p-1} \varepsilon m^{(p+1)p-1} |a| & \text{if } p < \infty \\ \varepsilon m |a| & \text{if } p = \infty. \end{cases}$$

Secondly, there exists a constant L , depending only on p , such that at all times t we have

$$(2.2) \quad ||PR^t(a, 0)|| \leq \begin{cases} L \varepsilon m^{(p+1)p-1} |a| & \text{if } p < \infty \\ L \varepsilon m |a| & \text{if } p = \infty \end{cases}$$

and if $t \leq m$ then

$$(2.3) \quad ||PR^t(a, 0)|| \leq \begin{cases} L \varepsilon m^{p-1} t |a| & \text{if } p < \infty \\ L \varepsilon t |a| & \text{if } p = \infty. \end{cases}$$

Proof. We estimate the norm of the sum of the standing and moving bumps. If $p = \infty$ we simply measure the absolute height of the sum of the bumps to obtain the values listed above, with $L = 1$. From now on, we shall assume that $p < \infty$. Set

$$L = \left(\frac{2^{p+3}}{p+1}\right)^{\frac{1}{p}} > \left(2 + \frac{2^{p+2} + 1}{p+1}\right)^{\frac{1}{p}}.$$

For (2.1), we have

$$||PR^t(a, 0)||^p \geq 2\varepsilon^p |a|^p \int_0^m s^p \, ds = \frac{2\varepsilon^p |a|^p}{p+1} m^{p+1}.$$

To establish (2.2), we note that the maximum value of the norm is attained when the supports of the standing and moving bumps are disjoint, which occurs if and only if $2m \leq t \leq T - 2m$. Thus we estimate

$$||PR^t(a, 0)||^p \leq 4\varepsilon^p |a|^p \int_0^{m+1} s^p \, ds = \frac{4\varepsilon^p |a|^p}{p+1} (m+1)^{p+1} \leq \frac{2^{p+3} \varepsilon^p |a|^p}{p+1} m^{p+1}.$$

For (2.3), when $t \leq m$, we note

$$\begin{aligned} ||PR^t(a, 0)||^p &\leq 2\varepsilon^p |a|^p \left\{ (m-t)t^p + \int_0^{t+1} s^p \, ds + \int_0^{\frac{t}{2}} (2s)^p \, ds \right\} \\ &= 2\varepsilon^p |a|^p \left\{ (m-t)t^p + \frac{(t+1)^{p+1}}{p+1} + \frac{t^{p+1}}{2(p+1)} \right\} \end{aligned}$$

$$\leq \left(2 + \frac{2^{p+2} + 1}{p + 1}\right) \varepsilon^p m t^p |a|^p.$$

□

In order to build our operator R on a Banach space X with a symmetric basis, we will need to estimate the norms of certain vectors in X with reasonable precision. In order to do this, we combine the estimates of Lemma 2.1 with a result closely based on a theorem of Tzafriri [9]. We have altered the statement of the next result to suit our purposes. In what follows, $d(\cdot, \cdot)$ indicates Banach-Mazur distance and, as above, $[\cdot]$ denotes norm-closed linear span.

Proposition 2.2 ([9, Proposition 5]). *Let V be a 2^n -dimensional vector space with basis $(v_\sigma)_{\sigma \in G}$, where G is the set of all functions from $\{1, \dots, n\}$ to $\{-1, 1\}$. Suppose that there are constants $K > 0$ and $r > 2$ such that given scalars a_σ , $\sigma \in G$, we have*

$$\frac{K^{-1}}{(2^n)^{\frac{1}{s}}} \left(\sum_{\sigma \in G} |a_\sigma|^s \right)^{\frac{1}{s}} \leq \left\| \sum_{\sigma \in G} a_\sigma v_\sigma \right\| / \left\| \sum_{\sigma \in G} v_\sigma \right\| \leq \frac{K}{(2^n)^{\frac{1}{r}}} \left(\sum_{\sigma \in G} |a_\sigma|^r \right)^{\frac{1}{r}}$$

where $r^{-1} + s^{-1} = 1$. Then there exists M , dependent on K and r , but independent of V and n , with the property that if we define

$$z_l = \sum_{\sigma \in G} \sigma(l) v_\sigma$$

for $1 \leq l \leq n$, then $d([z_l]_{l=1}^n, \ell_2^n) < M$.

The proof of the next result closely follows that of [9, Theorem 1], although we note that the assumed symmetry of the norm allows us to bypass the Ramsey arguments which feature in [9]. Tzafriri's notation has also been modified slightly to suit our requirements.

Lemma 2.3. *Let X have a normalised symmetric basis $(e_i)_{i=1}^\infty$ with conjugate system $(e_i^*)_{i=1}^\infty$ and symmetric norm $\|\cdot\|$. Then there exists $M > 0$ and $p \in \{1, 2, \infty\}$, a pairwise disjoint family of finite subsets $F_n \subseteq \mathbb{N}$, $n \in \mathbb{N}$, vectors $z_{l,n}$, $1 \leq l \leq n$, supported on F_n and permutations π_n of F_n with three properties:*

- (1) *given n , if a linear operator S on X satisfies $S(e_i) = e_{\pi_n(i)}$ for all $i \in F_n$, then $S(z_{l,n}) = z_{\tau(l),n}$, where τ is the cycle $(1, \dots, n)$;*
- (2) *$d([z_{l,n}]_{l=1}^n, \ell_p^n) < M$ for all n ;*
- (3) *π_n has order n .*

Proof. Define

$$\lambda(n) = \|e_1 + \dots + e_n\| \quad \text{and} \quad \mu(n) = \|e_1^* + \dots + e_n^*\|.$$

We follow the proof of [9, Theorem 1] in distinguishing three cases.

Case I: for every $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ such that $\lambda(nm_n)/\lambda(m_n) < 2$. Put $p = \infty$. Set $k_1 = 0$ and, given k_n , define $k_{n+1} = k_n + nm_n$. Let

$$F_n = \{k_n + 1, \dots, k_n + nm_n\}$$

and define

$$z_{l,n} = (e_{k_n+(l-1)m_n+1} + \dots e_{k_n+lm_n})/\lambda(m_n)$$

for $1 \leq l \leq n$, $n \in \mathbb{N}$. Finally, define

$$\pi_n(k_n + (l-1)m_n + r) = \begin{cases} k_n + lm_n + r & \text{if } 1 \leq l < n \text{ and } 1 \leq r \leq m_n \\ k_n + r & \text{if } l = n \text{ and } 1 \leq r \leq m_n. \end{cases}$$

It is clear that the F_n are pairwise disjoint and properties (1) and (3) hold. Now we prove (2). By the symmetry of the norm, we have $\|z_{l,n}\| = 1$. Since

$$\begin{aligned} \max_{l=1}^n |a_l| &\leq \left\| \sum_{l=1}^n a_l z_{l,n} \right\| \leq \max_{l=1}^n |a_l| \left\| \sum_{l=1}^n z_{l,n} \right\| \\ &\leq \max_{l=1}^n |a_l| \frac{\lambda(nm_n)}{\lambda(m_n)} \leq 2 \max_{l=1}^n |a_l| \end{aligned}$$

for any scalars a_1, \dots, a_n , we can see that (2) holds for any $M > 2$.

Case II: for every $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ such that $\mu(nm_n)/\mu(m_n) < 2$. Now put $p = 1$ and set k_n , F_n and π_n exactly as in case I. If we set

$$z_{l,n}^* = (e_{k_n+(l-1)m_n+1}^* + \dots e_{k_n+lm_n}^*)/\mu(m_n).$$

then we have

$$\max_{l=1}^n |a_l| \leq \left\| \sum_{l=1}^n a_l z_{l,n}^* \right\| \leq 2 \max_{l=1}^n |a_l|$$

just as above. Let $z_{1,n}$ satisfy $\|z_{1,n}\| = 1$ and $z_{1,n}^*(z_{1,n}) \geq \frac{1}{2}$, and have support contained in $\{k_n+1, k_n+m_n\}$, i.e., the support of $z_{1,n}^*$. If we let S be a linear operator satisfying $S(e_i) = e_{\pi_n(i)}$ for $i \in F_n$, and define $z_{l,n} = S^{l-1}(z_{1,n})$ for $1 < l \leq n$, then it follows by the symmetry of the norm that $\|z_{l,n}\| = 1$ and $z_{l,n}^*(z_{l,n}) = z_{1,n}^*(z_{1,n})$ whenever $1 \leq l \leq n$. By design, we have ensured that (1) holds. To check (2), we observe that

$$\left\| \sum_{l=1}^n a_l z_{l,n} \right\| \leq \sum_{l=1}^n |a_l| \leq 2 \left(\sum_{l=1}^n \lambda_l z_{l,n}^* \right) \left(\sum_{k=1}^n a_k z_{k,n} \right) \leq 4 \left\| \sum_{l=1}^n a_l z_{l,n} \right\|$$

where $\lambda_l a_l = |a_l|$ for $1 \leq l \leq n$. Therefore (2) holds whenever $M > 4$.

Case III: if neither case I nor case II hold then, following the proof of [9, Theorem 1] in case III, we obtain constants $K > 0$ and $r > 2$ such that for all $n \in \mathbb{N}$ and scalars a_1, \dots, a_n , we have

$$\frac{K^{-1}}{n^{\frac{1}{s}}} \left(\sum_{l=1}^n |a_l|^s \right)^{\frac{1}{s}} \leq \frac{1}{\lambda(n)} \left\| \sum_{l=1}^n a_l e_{n+l} \right\| \leq \frac{K}{n^{\frac{1}{r}}} \left(\sum_{l=1}^n |a_l|^r \right)^{\frac{1}{r}}$$

where $r^{-1} + s^{-1} = 1$. We set $p = 2$ and

$$F_n = \{2^n + 1, \dots, 2^{n+1}\}.$$

Fix n and let f be a bijection from $F = F_n$ to G , where G is as in Proposition 2.2. Put $v_\sigma = e_{f^{-1}(\sigma)}$ for $\sigma \in G$, and let z_l , $1 \leq l \leq n$, be as in Proposition 2.2. Let τ be the cycle $(1, \dots, n)$, define a permutation $\hat{\pi}$ on G by $\hat{\pi}(\sigma) = \sigma \circ \tau^{-1}$, and then set

$\pi = f^{-1} \circ \hat{\pi} \circ f$. We have (3). If S is an operator on X satisfying $S(e_i) = e_{\pi(i)}$ then we calculate

$$\begin{aligned} S(z_l) &= S\left(\sum_{\sigma \in G} \sigma(l) v_\sigma\right) = S\left(\sum_{\sigma \in G} \sigma(l) e_{f^{-1}(\sigma)}\right) \\ &= \sum_{\sigma \in G} \sigma(l) e_{f^{-1}(\hat{\pi}(\sigma))} \\ &= \sum_{\sigma \in G} \sigma(l) v_{\hat{\pi}(\sigma)} = \sum_{\sigma \in G} (\sigma \circ \tau)(l) v_\sigma = z_{\tau(l)}. \end{aligned}$$

Moreover, by construction, we have ensured that $d([z_l]_{l=1}^n, \ell_p^n) < M$. \square

We remark that we can follow the proof of [9, Theorem 1] a little more to show that the subspaces $[z_l]_{l=1}^n$, $n \in \mathbb{N}$, are uniformly complemented in X , that is, they are the images of a sequence of projections which are uniformly bounded in norm. However, we do not require this particular property of the $[z_l]_{l=1}^n$.

3. PROOF OF THEOREM 1.2

Let X , Y , E and J be as in Theorem 1.2. As X has a symmetric basis, it is isomorphic to its closed, finite-codimensional subspaces. Moreover, X is isomorphic to the space X_2^{d-1} , which denotes the product X^{d-1} , endowed with the norm given by $\|(x_1, \dots, x_{d-1})\|^2 = \sum_{j=1}^{d-1} \|x_j\|^2$. Therefore, by considering a suitable isomorphism, Theorem 1.2 follows from the following proposition.

Proposition 3.1. *Whenever E is a closed, symmetric subset of $S_{\ell_2^d}$, with $2 \leq d < \infty$, then there exists an operator $R : \ell_2^d \oplus X_2^{d-1} \longrightarrow \ell_2^d \oplus X_2^{d-1}$ with two properties:*

- (1) *if $u \in E$ then $\|R^n(u, x)\| \rightarrow \infty$;*
- (2) *if $u \in S_{\ell_2^d} \setminus E$ then there is a subsequence $(R^{n_i}(u, x))$ of $(R^n(u, x))$ such that $R^{n_i}(u, x) \rightarrow (u, x)$ weakly.*

We shall prove Proposition 3.1 with a sequence of lemmas. The proofs in the real and complex cases are practically identical. First, we consider a map ρ defined on $S_{\ell_2^d}$ by

$$\rho(u, v)^2 = 1 - |\langle u, v \rangle|^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product. This function ρ is a pseudometric on $S_{\ell_2^d}$ which, given $u \in S_{\ell_2^d}$, identifies the points λu , $|\lambda| = 1$. We shall also define

$$\rho(u, E) = \inf\{\rho(u, v) : v \in E\}.$$

Since E is closed and symmetric, it follows that $\rho(u, E) = 0$ if and only if $u \in E$. Given $v \in S_{\ell_2^d} \setminus E$, we select an orthonormal basis $e_{v,1}, \dots, e_{v,d-1}$ of the perpendicular subspace v^\perp and define $\Delta_v : \ell_2^d \longrightarrow \ell_2^{d-1}$ by

$$\Delta_v(u) = \frac{1}{\rho(v, E)} (\langle u, e_{v,1} \rangle, \dots, \langle u, e_{v,d-1} \rangle).$$

Obviously, $\|\Delta_v(u)\|_2 = \rho(u, v)/\rho(v, E)$ whenever $u \in S_{\ell_2^d}$, where $\|\cdot\|_2$ denotes the usual norm on ℓ_2^{d-1} .

Let $W_n = \{v \in S_{\ell_2^d} : \rho(v, E) \geq 2^{-n}\}$. It is a straightforward matter to show that for each n , we can find a 2^{-n} -net of $S_{\ell_2^d}$, with respect to ρ , which has size of order 2^{2d-1} (or 2^{d-1} if we are considering real Banach spaces). Therefore, there exists an integer K such that, for each n , there exist vectors

$$v_1^n, \dots, v_{K2^{n(2d-1)}}^n \in W_n,$$

with repetitions if necessary, with the property that for any $u \in W_n$, we can find v_i^n satisfying $\rho(u, v_i^n) \leq 2^{-n}$.

Lemma 3.2. *Let $u \in S_{\ell_2^d}$. Firstly*

$$(3.1) \quad \|\Delta_v(u)\|_2 \leq 2^{-n} \quad \text{whenever } v \in W_n.$$

Secondly, if $u \in E$ then

$$(3.2) \quad \|\Delta_v(u)\|_2 \geq 1 \quad \text{whenever } v \in S_{\ell_2^d} \setminus E.$$

Finally, if $u \notin E$ then there exists n_0 with the property that whenever $n > n_0$, there exists i in the range $1 \leq i \leq K2^{n(2d-1)}$, such that

$$(3.3) \quad \|\Delta_{v_i^n}(u)\|_2 \leq 2^{n_0+1-n}.$$

Proof. We only prove (3.3). Fix n_0 such that $u \in W_{n_0}$. For $n > n_0$, we can find $v = v_i^n$ such that $\rho(u, v) \leq 2^{-n}$. Therefore

$$\rho(v, E) \geq \rho(u, E) - \rho(u, v) \geq 2^{-n_0} - 2^{-n} \geq 2^{-(n_0+1)}$$

and $\|\Delta_v(u)\|_2 \leq 2^{n_0+1-n}$. □

We take constants $m_k, T_k \in \mathbb{N}$ and $\varepsilon_k > 0$. The values of these constants will be chosen in due course. Let $M, p, F_n, z_{l,n}$ and π_n be as in Lemma 2.3. Define

$$S(e_i) = \begin{cases} e_{\pi_{T_k}(i)} & \text{if } i \in F_{T_k} \text{ for some } T_k \\ e_i & \text{otherwise} \end{cases}$$

and extend S linearly to X . As $\|\cdot\|$ is symmetric, S is an isometry. Define operators $S_k : [z_{l,T_k}]_{l=1}^{T_k} \longrightarrow [z_{l,T_k}]_{l=1}^{T_k}$ and $F_k : \mathbb{R} \longrightarrow [z_{l,T_k}]_{l=1}^{T_k}$ by

$$S_k \left(\sum_{l=1}^{T_k} y_l z_{l,T_k} \right) = \sum_{l=1}^{T_k} y_l z_{\tau(l), T_k}$$

where τ is the cycle $(1, \dots, T_k)$, and

$$F_k(a) = a\varepsilon_k \sum_{l=1}^{m_k} z_{l,T_k} - a\varepsilon_k \sum_{l=m_k+1}^{2m_k} z_{l,T_k}.$$

Then define R_k on $\mathbb{R} \oplus [z_{l,T_k}]_{l=1}^{T_k}$ by

$$R_k(a, y) = (a, S_k(y) + F_k(a))$$

and let $P_k(a, y) = y$ for $a \in \mathbb{R}$ and $y \in [z_{l,T_k}]_{l=1}^{T_k}$. Let Q and Q_j be the standard projections of $\ell_2^d \oplus X_2^{d-1}$ onto ℓ_2^d and onto the j th copy of X , respectively. We

define integers $C_1 = 1$ and $C_{n+1} = C_n + K2^{n(2d-1)}$ for $n \geq 1$, and set $w_k = v_{k+1-C_n}^n$ whenever $C_n \leq k < C_{n+1}$.

Finally, if $u \in \ell_2^d$ and $x \in X_2^{d-1}$, we can define an operator R on $\ell_2^d \oplus X_2^{d-1}$ by

$$QR(u, x) = u \quad \text{and} \quad Q_j R(u, x) = SQ_j(0, x) + \sum_{k=1}^{\infty} \frac{F_k(\langle u, e_{w_k, j} \rangle)}{\rho(w_k, E)}$$

for $1 \leq j \leq d-1$ and where $e_{w_k, j}$ is the j th element of the given basis of w_k^\perp , chosen above.

Of course, it is necessary to choose the constants m_k, T_k and ε_k so that R is bounded and maps into X . First let $m_1 = 1$ and $T_0 = 1$. Then set $T_k = (5^{dn} + 1)T_{k-1}$, $m_k = T_{k-1} - m_{k-1}$ and

$$\varepsilon_k = \begin{cases} \frac{n}{m_k^{(p+1)p^{-1}}} & \text{if } p = 1 \text{ or } p = 2 \\ \frac{n}{m_k} & \text{if } p = \infty \end{cases}$$

whenever $C_n \leq k < C_{n+1}$. Our first task is to show that, with respect to these constants, R is a bounded operator mapping into $\ell_2^d \oplus X_2^{d-1}$.

Lemma 3.3. *The operator R is bounded and maps into $\ell_2^d \oplus X_2^{d-1}$.*

Proof. It is enough to show that $\sum_{k=1}^{\infty} \rho(w_k, E)^{-1} F_k(\langle u, e_{w_k, j} \rangle)$ is absolutely summable for $1 \leq j \leq d-1$. Assume $u \in S_{\ell_2^d}$. By Lemma 2.3, part 2, we have

$$(3.4) \quad M^{-\frac{1}{2}} \|y\|_p \leq \left\| \sum_{l=1}^{T_k} y_l z_{l, T_k} \right\| \leq M^{\frac{1}{2}} \|y\|_p$$

where $y = (y_1, \dots, y_{T_k}) \in \ell_p^{T_k}$ and $\|\cdot\|_p$ is the usual norm on $\ell_p^{T_k}$. We shall assume that $M \geq L$, where L is as in Lemma 2.1. Note that $F_k(a) = S_k(0) + F_k(a) = P_k R_k(a, 0)$. Therefore, from (2.3) with $t = 1$, (3.1), (3.4) and the definition of ε_k , we have

$$\begin{aligned} \frac{\|F_k(\langle u, e_{w_k, j} \rangle)\|}{\rho(w_k, E)} &\leq \begin{cases} M^{\frac{3}{2}} \varepsilon_k m_k^{p-1} \rho(w_k, E)^{-1} |\langle u, e_{w_k, j} \rangle| & \text{if } p = 1, 2 \\ M^{\frac{3}{2}} \varepsilon_k \rho(w_k, E)^{-1} |\langle u, e_{w_k, j} \rangle| & \text{if } p = \infty \end{cases} \\ &\leq M^{\frac{3}{2}} n \|\Delta_{w_k}(u)\|_2 m_k^{-1} \\ &\leq M^{\frac{3}{2}} n 2^n m_k^{-1} \end{aligned}$$

whenever $C_n \leq k < C_{n+1}$.

From the definitions of m_k and T_k , we obtain

$$(3.5) \quad m_{k+1} = T_k - m_k = T_k - T_{k-1} + m_{k-1} \geq 5^{dn} T_{k-1} \geq 5^{dn} m_k$$

whenever $C_n \leq k < C_{n+1}$. In particular, $m_{k+1} \geq 5^d m_k$. Therefore

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\|F_k(\langle u, e_{w_k, j} \rangle)\|}{\rho(w_k, E)} &\leq \sum_{n=1}^{\infty} \sum_{k=C_n}^{C_{n+1}-1} M^{\frac{3}{2}} n 2^n m_k^{-1} \\ &\leq M^{\frac{3}{2}} m_1^{-1} \sum_{n=1}^{\infty} \sum_{k=C_n}^{C_{n+1}-1} \frac{n 2^n}{5^{d(k-1)}} \end{aligned}$$

$$\begin{aligned}
&\leq M^{\frac{3}{2}}5^d \sum_{n=1}^{\infty} \sum_{k=C_n}^{C_{n+1}-1} \frac{n2^n}{5^{dn}} \\
&= M^{\frac{3}{2}}5^d K \sum_{n=1}^{\infty} 2^{n(2d-1)} \frac{n2^n}{5^{dn}} = M^{\frac{3}{2}}5^d K \sum_{n=1}^{\infty} n\left(\frac{4}{5}\right)^{dn}
\end{aligned}$$

bearing in mind that $k-1 \geq C_n-1 \geq n-1$ whenever $C_n \leq k < C_{n+1}$. Hence R is bounded and $R(u, x) \in \ell_2^d \oplus X_2^{d-1}$. \square

In order to analyse the behaviour of $R^m(u, x)$, it will help to consider separately $R^m(u, 0)$ and $R^m(0, x)$.

Lemma 3.4. *Given $(u, x) \in \ell_2^d \oplus X_2^{d-1}$, we have*

$$(3.6) \quad Q_j R^m(0, x) = S^m Q_j(0, x)$$

and

$$(3.7) \quad Q_j R^m(u, 0) = \sum_{k=1}^{\infty} \frac{P_k R_k^m(\langle u, e_{w_k, j} \rangle, 0)}{\rho(w_k, E)}$$

for $1 \leq j \leq d-1$ and for all m .

Proof. We proceed by induction. Clearly $Q_j R(0, x) = S Q_j(0, x)$, so (3.6) holds for $m = 1$. If (3.6) holds for some $m \geq 1$ and $R^m(0, x) = (0, y)$ then

$$Q_j R^{m+1}(0, x) = Q_j R(0, y) = S Q_j(0, y) = S Q_j R^m(0, x) = S^{m+1} Q_j(0, x).$$

Now

$$P_k R_k(a, 0) = S_k(0) + F_k(a) = F_k(a)$$

and $S Q_j(u, 0) = 0$, so (3.7) holds for $m = 1$. Assume that (3.7) holds for some $m \geq 1$. Suppose that

$$P_k R_k^m(a, 0) = y = \sum_{l=1}^{T_k} y_l z_{l, T_k}.$$

By Lemma 2.3, we have ensured that $S(y) = S_k(y)$. Furthermore, we observe

$$\begin{aligned}
P_k R_k^{m+1}(a, 0) &= P_k R_k(R_k^m(a, 0)) = P_k R_k(a, y) \\
&= S_k(y) + F_k(a) \\
&= S(y) + F_k(a) = S P_k R_k^m(a, 0) + F_k(a).
\end{aligned}$$

Therefore, if $R^m(u, 0) = (u, z)$ then

$$\begin{aligned}
Q_j R^{m+1}(u, 0) &= Q_j R(u, z) \\
&= S Q_j(0, z) + \sum_{k=1}^{\infty} \frac{F_k(\langle u, e_{w_k, j} \rangle)}{\rho(w_k, E)} \\
&= S Q_j R^m(u, 0) + \sum_{k=1}^{\infty} \frac{F_k(\langle u, e_{w_k, j} \rangle)}{\rho(w_k, E)}
\end{aligned}$$

$$\begin{aligned}
&= S \left(\sum_{k=1}^{\infty} \frac{P_k R_k^m(\langle u, e_{w_k, j} \rangle, 0)}{\rho(w_k, E)} \right) + \sum_{k=1}^{\infty} \frac{F_k(\langle u, e_{w_k, j} \rangle)}{\rho(w_k, E)} \\
&= \sum_{k=1}^{\infty} \frac{S P_k R_k^m(\langle u, e_{w_k, j} \rangle, 0) + F_k(\langle u, e_{w_k, j} \rangle)}{\rho(w_k, E)} \\
&= \sum_{k=1}^{\infty} \frac{P_k R_k^{m+1}(\langle u, e_{w_k, j} \rangle, 0)}{\rho(w_k, E)}
\end{aligned}$$

as required. \square

The consequence of Lemma 3.4 is that we can split the analysis of $R^m(u, x)$ into two parts: the ‘shift’ and the ‘perturbation’. First, we examine the behaviour of the shift.

Lemma 3.5. *Given $x \in X_2^{d-1}$, we have $\|R^m(0, x)\| = \|(0, x)\|$ for all m . Moreover, $R^{T_k}(0, x) \xrightarrow{w} (0, x)$.*

Proof. Given (3.6) and the fact that S is an isometry, the first assertion is trivial. Now consider the weak convergence. Let $f \in X^*$ with $\|f\| = 1$, $\varepsilon > 0$ and $1 \leq j \leq d-1$. If $Q_j(0, x) = \sum_{i=1}^{\infty} x_i e_i$, we take $k \in \mathbb{N}$ such that

$$\left\| \sum_{l=k+1}^{\infty} \sum_{i \in F_{T_l}} x_i e_i \right\| < \varepsilon.$$

Since T_l divides T_r whenever $l \leq r$, we can see that $\pi_{T_l}^{T_r}$ is the identity for such l . Therefore, if $r \geq k$, we estimate

$$\begin{aligned}
|f(Q_j R^{T_k}(0, x) - Q_j(0, x))| &= |f(S^{T_r} Q_j(0, x) - Q_j(0, x))| \\
&= \left| f \left(\sum_{l=r+1}^{\infty} \sum_{i \in F_{T_l}} x_i e_{\pi_{T_l}^{T_r}(i)} - \sum_{l=r+1}^{\infty} \sum_{i \in F_{T_l}} x_i e_i \right) \right| \\
&\leq 2 \left\| \sum_{l=r+1}^{\infty} \sum_{i \in F_{T_l}} x_i e_i \right\| \\
&\leq 2 \left\| \sum_{l=k+1}^{\infty} \sum_{i \in F_{T_l}} x_i e_i \right\| < 2\varepsilon
\end{aligned}$$

by symmetry of the norm. Thus $Q_j R^{T_k}(0, x) \xrightarrow{w} Q_j(0, x)$ whenever $1 \leq j \leq d-1$. The weak convergence of $R^{T_k}(0, x)$ to $(0, x)$ follows. \square

Now we analyse the behaviour of the perturbation. Ultimately, it is the perturbation that drives the behaviour of the system as a whole.

Lemma 3.6. *If $u \in E$ then*

$$\|R^m(u, 0)\| \rightarrow \infty.$$

On the other hand, if $u \in S_{\ell_2^d} \setminus E$ then there exists k_n in the range $C_n \leq k_n < C_{n+1}$ with the property that

$$\|R^{T_{k_n-1}}(u, 0) - (u, 0)\| \rightarrow 0.$$

Proof. Let $u \in E$ and suppose that $p = 1$ or $p = 2$. Assume that $m_k \leq m < T_k - m_k = m_{k+1}$ and $C_n \leq k < C_{n+1}$. Then by (2.1), (3.2), (3.4), (3.7) and the definition of ε_k , we have

$$\begin{aligned} \|R^m(u, 0)\|^2 &\geq \sum_{j=1}^{d-1} \|Q_j R^m(u, 0)\|^2 \\ &\geq \sum_{j=1}^{d-1} \left(\frac{\|P_k R_k^m(\langle u, e_{w_k, j} \rangle, 0)\|}{\rho(w_k, E)} \right)^2 \\ &\geq \sum_{j=1}^{d-1} M^{-1} \left(\frac{2}{p+1} \right)^{2p-1} \varepsilon_k^2 m_k^{2(p+1)p-1} \left(\frac{|\langle u, e_{w_k, j} \rangle|}{\rho(w_k, E)} \right)^2 \\ &= M^{-1} \left(\frac{2}{p+1} \right)^{2p-1} n^2 \sum_{j=1}^{d-1} \left(\frac{|\langle u, e_{w_k, j} \rangle|}{\rho(w_k, E)} \right)^2 \\ &\geq M^{-1} \left(\frac{2}{p+1} \right)^{2p-1} n^2 \end{aligned}$$

If $u \in E$ and $p = \infty$ then similarly, we obtain

$$\|R^m(u, 0)\|^2 \geq M^{-1} n^2.$$

Either way, $\|R^m(u, 0)\| \rightarrow \infty$.

Instead, if $u \in S_{\ell_2^d} \setminus E$ then by (3.3), for every $n > n_0$ there exists k_n in the range $C_n \leq k_n < C_{n+1}$, such that

$$\|\Delta_{w_{k_n}}(u)\|_2 \leq 2^{n_0+1-n}$$

By (2.2), (3.4) and the definition of ε_k , we have

$$\begin{aligned} \left\| P_{k_n} R_{k_n}^{T_{k_n-1}}(\langle u, e_{w_{k_n}, j} \rangle, 0) \right\| &\leq \begin{cases} M^{\frac{3}{2}} \varepsilon_{k_n} m_{k_n}^{(p+1)p-1} |\langle u, e_{w_{k_n}, j} \rangle| & \text{if } p = 1, 2 \\ M^{\frac{3}{2}} \varepsilon_{k_n} m_{k_n} |\langle u, e_{w_{k_n}, j} \rangle| & \text{if } p = \infty \end{cases} \\ (3.8) \quad &\leq M^{\frac{3}{2}} \rho(w_{k_n}, E) n 2^{n_0+1-n}. \end{aligned}$$

Then we notice that if $r \leq k_n - 1$, we have

$$(3.9) \quad \left\| P_r R_r^{T_{k_n-1}}(\langle u, e_{w_r, j} \rangle, 0) \right\| = 0$$

because $R_r^{T_r}$ is the identity and T_r divides T_{k_n-1} whenever $r \leq k_n - 1$. Now we have to estimate $\left\| P_r R_r^{T_{k_n-1}}(\langle u, e_{w_r, j} \rangle, 0) \right\|$ for $r \geq k_n + 1$. If $r \geq k_n + 1$ then from (3.5), we have

$$m_r \geq 5^{d(r-(k_n+1))} m_{k_n+1} \geq 5^{d(r-(k_n+1))} 5^{dn} T_{k_n-1}.$$

Take $l \geq n$ such that $C_l \leq r < C_{l+1}$. Again we assume that $M \geq L$, with L as in Lemma 2.1. We apply (2.3), (3.1) and (3.4) to obtain

$$\begin{aligned}
(3.10) \quad & \left\| P_r R_r^{T_{k_n-1}}(\langle u, e_{w_r, j} \rangle, 0) \right\| \\
& \leq \begin{cases} M^{\frac{3}{2}} \varepsilon_r m_r^{p-1} T_{k_n-1} |\langle u, e_{w_r, j} \rangle| & \text{if } p = 1, 2 \\ M^{\frac{3}{2}} \varepsilon_r T_{k_n-1} |\langle u, e_{w_r, j} \rangle| & \text{if } p = \infty \end{cases} \\
& \leq M^{\frac{3}{2}} T_{k_n-1} \frac{\rho(w_r, E) l 2^l}{m_r} \\
& \leq M^{\frac{3}{2}} \rho(w_r, E) \frac{l 2^l}{5^{d(r-(k_n+1))} 5^{dn}} \\
& \leq M^{\frac{3}{2}} \rho(w_r, E) \frac{l 2^l}{5^{dn} 5^{d(l-n)}} \quad \text{since } l - n \leq r - (k_n + 1) \\
& = M^{\frac{3}{2}} \rho(w_r, E) \frac{l 2^l}{5^{dl}}
\end{aligned}$$

Combining (3.7) with (3.8), (3.9) and (3.10) gives

$$\begin{aligned}
\|Q_j R^{T_{k_n-1}}(u, 0)\| & \leq \sum_{r=1}^{\infty} \frac{\left\| P_r R_r^{T_{k_n-1}}(\langle u, e_{w_r, j} \rangle, 0) \right\|}{\rho(w_r, E)} \\
& = \sum_{r=k_n}^{\infty} \frac{\left\| P_r R_r^{T_{k_n-1}}(\langle u, e_{w_r, j} \rangle, 0) \right\|}{\rho(w_r, E)} \\
& = M^{\frac{3}{2}} n 2^{n_0+1-n} + \sum_{r=k_n+1}^{\infty} \frac{\left\| P_r R_r^{T_{k_n-1}}(\langle u, e_{w_r, j} \rangle, 0) \right\|}{\rho(w_r, E)} \\
& \leq M^{\frac{3}{2}} n 2^{n_0+1-n} + M^{\frac{3}{2}} \sum_{l=n}^{\infty} K 2^{l(2d-1)} \frac{l 2^l}{5^{dl}} \\
& = M^{\frac{3}{2}} n 2^{n_0+1-n} + M^{\frac{3}{2}} K \sum_{l=n}^{\infty} l \left(\frac{4}{5}\right)^{dl} \\
& \rightarrow 0
\end{aligned}$$

as $n \rightarrow \infty$. This concludes the proof. \square

Proof of Proposition 3.1. Let $u \in E$. By Lemmas 3.5 and 3.6 we have

$$\|R^m(u, x)\| \geq \|R^m(u, 0)\| - \|R^m(0, x)\| = \|R^m(u, 0)\| - \|(0, x)\| \rightarrow \infty$$

as $m \rightarrow \infty$.

Now suppose $u \in S_{\ell_2^d} \setminus E$. Again by Lemmas 3.5 and 3.6, we can pick suitable k_n such that

$$R^{T_{k_n-1}}(u, x) = R^{T_{k_n-1}}(u, 0) + R^{T_{k_n-1}}(0, x) \xrightarrow{w} (u, 0) + (0, x) = (u, x).$$

\square

If $X = c_0$ or $X = \ell_p$, $1 \leq p < \infty$, then we can simplify the proof of Theorem 1.2 by replacing the $z_{l,n}$ with unit vectors and replacing the corresponding π_n with cycles. Since there is a Banach space with a symmetric basis but containing no isomorphic copy of c_0 or ℓ_p , $1 \leq p < \infty$, [3], it is not possible to obtain Theorem 1.2 by proving it in the cases $X = c_0$ and $X = \ell_p$, and then appealing to complemented subspaces.

4. PROBLEMS

Since the operators constructed in this note rely fundamentally on permutations of basis vectors, it makes sense to pose the following question.

Problem 4.1. *If X is a Banach space with an unconditional basis, does there exist an operator T on X with the property that $\|T^n(x)\| \rightarrow \infty$ for some $x \in X$, and $\|T^n y\| \not\rightarrow \infty$ for all y in a non-empty, open subset of X ?*

Also, since the operators which feature above cannot be compact, by Proposition 1.3, the next question seems natural to us.

Problem 4.2. *Is it possible to find a sum $I + T$, where T is compact, which satisfies the properties given in the abstract? In particular, does the Argyros-Haydon space [1] admit such an operator?*

If no sum $I + T$, T compact, satisfies the properties given in the abstract, then this suggests to us that some kind of unconditional structure is necessary in order to construct such operators.

Finally, we make a remark about the title of this note. The operator R constructed above can be viewed as a machine which acts on a countable family of disjoint cycles. This family of disjoint cycles can be seen as a countable directed graph. We speculate that it may be possible to construct operators with other interesting properties by basing them on more complicated directed graphs.

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